STUDY OF NEAR RING STRUCTURE FOR MATHEMATICAL APPLICATIONS

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Abstract**—** In mathematics, a near-ring (also near ring or nearring) is an algebraic structure similar to a ring but satisfying fewer axioms. Near-rings arise naturally from functions on groups. A set N together with two binary operations + (called addition) and · (called multiplication) is called a (right) near-ring if:

A1: N is a group (not necessarily abelian) under addition;

A2: multiplication is associative (so N is a semigroup under multiplication); and

A3: multiplication on the right distributes over addition: for any x, y, z in N, it holds that (x + y)·z = (x·z) + (y·z).[1]

Similarly, it is possible to define a left near-ring by replacing the right distributive law A3 by the corresponding left distributive law. Both right and left near-rings occur in the literature; for instance, the book of Pilz[2] uses right near-rings, while the book of Clay[3] uses left near-rings.

An immediate consequence of this one-sided distributive law is that it is true that 0·x = 0 but it is not necessarily true that x·0 = 0 for any x in N. Another immediate consequence is that (−x)·y = −(x·y) for any x, y in N, but it is not necessary that x·(−y) = −(x·y). A near-ring is a ring (not necessarily with unity) if and only if addition is commutative and multiplication is also distributive over addition on the left. If the near-ring is with unity (that is, there exists a multiplicative identity), then distributive on both sides is sufficient, and commutatively of addition follows automatically.

Keywords – Multiplication, Mathematics, literature

I INTRODUCTION

A right near-ring (N, +, .) is an algebraic system with two binary operations such that (i) (N, +) is a group-not necessarily abelian –with 0 as its identity element, (ii) (N, .) is a semi-group (We write xy for x.y for all x,y in N) and (iii) (x+y)z = xz + yz for all x,y,z in N. Because of (iii) 0n = 0 for all n in N. As we do not stipulate the left distribute law, "n0 = 0" need not hold good for all n in N. We say that N is zero – symmetric if n0 =0 for all n in N. N is called an S-near-ring or an S'-near-ring according as x □ Nx or x □ xN for all x□ N. A subgroup M of N is called an N-subgroup if NM □ M and an invariant N-subgroup if, in addition , MN □ M.

An ideal I of N is called a semiprime ideal if for all ideals J of N, J 2 □ I □□□J□□□ I. If {0} is a semiprime ideal, then N is called a semiprime near-ring. An ideal I of N is called completely semiprime if x□ I whenever x2□ I. N is called a strictly prime near-ring if {0} is a strictly prime ideal i.e. if A and B are N-subgroups of N such that AB = {0}, then either A= {0} or B= {0}.

The concept of a mate function in N has been introduced in [4] with a view to handle the regularity structure in a near-ring with considerable case. A map m from N into N is called a mate function for N, if x= xm(x)x for all x in N. m(x) is called a mate of x.

Basic concepts and terms used but not defined in this paper can be found in Pilz [3]. Throughout this paper N stands for a near-ring – more precisely a right near-ring – with at least two elements.

As in P.249 Pilz[3], "if N is a near-field then either N is isomorphic to Mc(Z2) or N is zero-symmetric "(For the concept of Mc(Z2) one may refer to Example 1.4 (a), P.8 and 1.5, P.12 of Pilz [3]. Obviously Mc(Z2) is a near-field of order 2and is not zero –symmetric). All the near-fields in this paper are zero – symmetric.

Notations

(a) E denotes the set of all idempotents of N.
(b) L is the set of all nilpotent elements of N.
(c) Nd = {n□N/ n(x+y) = nx+ny, for all x,y□ N}-the set of all distributive elements of N.
(d) N0 = {n□N/ n0 = 0} – the zero- symmetric part of N(It is worth noting that N is zero –symmetric if N = N0}.
(e) C (N) = {n□N/ nx = xn for all x□N}.

II WEAKENED FORMS OF P (1, 2) AND P (2, 1) NEAR – RINGS

In this chapter we introduce the concepts of Weak P(1,2) and Weak P(2,1) near- rings.We say that N is Weak P(1,2) (Weak P(2,1)) if xN = yN

Nx2 = Ny2 for x,y in N (Nx = Ny □ x2N = y2N for x,y in N) . We give examples to show that each of these concepts is different in general. Weak P(1,2) (Weak P(2,1)) is a generalization of P(1,2) (P(2,1)).

We obtain characterization of Weak P(1,2) near-rings which admit mate functions. We show that the concepts of Weak P(1,2) , Weak P(2,1), P(1,2) and P(2,1) are all equivalent in a Ring with mate functions. We obtain a necessary and sufficient condition for a Weak P(1,2) near- ring to be a P(1,2) near- ring. We prove that the concept of Weak P(1,2) is preserved under homomorphisms and also obtain a structure theorem for Weak P(1,2) near – rings. Towards the end of this chapter we discuss briefly the relation between the concepts of Weak P(1,2) and Weak P(2,1) in a near – ring.

Definition 3.1.1.

We define N to be a Weak P(1,2) (Weak P(2,1)) if)) if xN = yN □

Nx2 = Ny2 (Nx = Ny □ x2N = y2N for x,y in N).

Remarks 3.1.2.

The motivation for such a generalization actually stems from the fact that the usual normality condition of a subgroup H of a group (G, .),namely gH = Hg for all g□G , is equivalent to either of the following weakened conditions

(i) g1H = g2H □ Hg1 = Hg2

(ii) Hg1 = Hg2 □ g1H = g2H for all g1,g2□G

Weak P(1,2) and Weak P(2,1) are two different concepts and neither implies the other in general. Also neither of them nor their combination will imply P(1,2) or P(2,1) in general. But it is obvious that when N is P(1,2) (P(2,1)) it is Weak P(1,2) (Weak P(2,1)).

The following examples justify these remarks.

Examples 3.1.3.

A near - field and direct product of ant two near – fields are naturally both Weak P(1,2) and Weak P(2,1) near-rings.

Any constant near- ring N is Weak P(1,2) but not Weak P(2,1).

Let (N, +) be the familiar group of integers modulo 5. We define „.‟ in N as follows as per scheme 7, p.408 of pilz [27].

B_1 NEAR-RINGS

In this chapter, we introduce the concepts of B1 near- rings, unit B1 near-rings and strong B1 near- rings. It includes three sections. Motivated by the concept of left bipotent near- rings, we define N to be a B1 near-ring if for every a□N, there exists x□N* such that Nax = Nxa. Motivated by the concept of unit regular near- rings , we define N to be a unit B1 near- ring if for every a□N, there exists a unit u□N such that Nau = Nua. By generalizing the concept of B1 near-rings, we define N to be a strong B1 near- ring if Nab = Nba for all a,b□N.

In the first section, we discuss some of the properties of B1 near- rings. We prove that certain near–rings are B1 near- rings. We obtain a condition under which every S2 near- ring is a B1 near- ring. We show that every ideal and every N- subgroup of a B1 near-ring without non- zero zero divisors is a B1 near- ring when it is a strong S1 near- ring.

In the second section, we introduce the concept of unit B1 near- rings. We show that every unit B1 near-ring is a B1 near- ring , but a B1 near- ring need not be a unit B1 near- ring. We prove that every strong S2 near-ring is an S near- ring, but an S near– ring need not be a strong S2 near- ring.

In the last section of this chapter, we disciuss the properties of strong B1 near- rings in detail. We prove that every strong B1 near- ring is a B1 near- ring, but a B1 near- ring need not be a strong B1 near- ring. We also show that every N-simple near- ring is a strong B1 near- ring, but a strong B1 near- ring need not be a N-simple near- ring. We show that every homomorphic image of a strong B1 near- ring is a strong B1 near- ring. We obtain a structure theorem for strong B1 near- rings and a condition for a strong S1 near-ring to be a strong

B1 near- ring. We show that every strong S2 near- ring is a strong B1 near- ring. We also obtain a characterization of strong B1 near- rings.

4.1 B1 near-rings

In this section, we define B1 near- rings and discuss some of the properties of such near- rings.

Definition 4.1.1.

We sat that N is a B1 near- ring if for every a□N, there exists x□N* such that Nax= Nxa.

Examples 4.1.2.

(a) Every constant near- ring is a B1 near- ring.

(b) Every commutative near- ring is a B1 near- ring.

(c) We consider the near- ring (Z4,+,.) where (Z4,+) is the group of integers modulo „4" and „." is defined as follows (scheme (4), p.407 of pilz [27]).

III PROPOSED WORK
THEORM ANALYSIS

Theorem 4.3.17.

Let N be a strong B1 near-ring. If N is Boolean then the following are true.

(i) NaNb = Nab for all a,b □ N.

(ii) All principal N-sub groups commute with one another.

(iii) Every ideal of N is a strong B1 near-ring.

(iv) Every N-subgroup of N is a strong B1 near-ring.

(v) Every N-subgroup of N is an invariant N-subgroup of N.

Proof:

Since N is a strong B1 near-ring, Nab = Nba for all a,b□N.

(i) Let a,b□N . Since N is Boolean, a = a2 □ aN. That is a□aN □ Na □ NaN□ Nab □ NaNb. Let y□NaNb. Then there exist n,n'□N such that y = nan'b. since N is a strong B1 near-ring, from Lemma 4.3.9 we get, nan1 = zn1a for some z□N. Therefore y = (zn'a)b = (zn')ab □ Nab. That is y □ Nab.

Therefore NaNb □ Nab. Thus NaNb = Nab.

(ii) Let a,b□N . Now NaNb = Nab [by (i)] = Nba =NbNa. Thus the desired result follows.

(iii) Let I be an ideal of N. Therefore IN C I. Let a,b Є I. Now Iab = Ia2b [since N is Boolean] = (Ia)ab □ (IN)ab = I(Nab) = I(Nba) = (IN)ba □ Iba. That is Iab □Iba. Similarly Iba □ Iab. Therefore Iab = Iba. Thus I is a strong B1 near-ring.

(iv) Let M be an N-subgroup of N. Therefore NM □ M. For any x,y□M, let z□Mxy [since M □N] = yx = Ny2x [since N is Boolean] = (Ny)yx □ (NM)yx □ Myx. That is z□Myx. Therefore Mxy □ Myx. Similarly Myx □ Mxy. Therefore Mxy = Myx. Thus M is a strong B1 near-ring.

(v) Let M be an N-subgroup of N. Therefore NM □ M. Let z□MN.Then there exist m□M and n□N such that z= mn = m2n [since N is Boolean]. That is z=m2n. Since N is a strong B1 near-ring, Lemma 4.3.9 demands that there exists n'□N such that m2n = n'nm. Therefore z = n'nm = (n'n)m□NM □ M.

Therefore MN □ M. Thus M is an invariantN-subgroup of N.

Theorem 4.3.18.

Let N be a strong B1 near-ring. If N =N0 is Boolean then the following are true:

(i) N has (*,IFP).

(ii) N has Property (P4).

(iii) N has strong IFP.

(iv) Every left ideal of N is an invariant N-subgroup of N.

(v) Every ideal is an invariant N-subgroup of N.

(vi) Every left ideal is an ideal.

Proof:

(i) Let x,y□N. Suppose xy =0 . Now from Lemma 4.3.9 we get, there exists n□N such that y2x = nxy = n0 = 0[since N =N0]. That is y2x = 0. Since N is Boolean, yx = 0. Again Lemma 4.3.9 demands that, there exists z□N such that x(xn)y = zy(xn). That is x2ny = zy(xn) = z(yx)n =z(0)n = 0 [since N = N0], Since N is Boolean, xny = 0. Thus N has (*, IFP).

(ii)Let x,y□N and let I be an ideal of N. Suppose xy □ I. Now Lemma 4.3.9 guarantees that there exists z□N such that y2x = zxy □ NI □ I [since N = N0]. That is y2x□I. Since N is Boolean, yx□I. Thus N has Property (P4).

(iii) Let x,y,n□N and let I be an ideal of N. Suppose xy□I. Form part (ii) we get, there exists z□N such that x2ny = x(xn)y = zy(xn) = z(yx)n □ NIN [by (ii)] C I.

That is x2ny□I. Since N is Boolean, xny□I. Thus N has strong IFP.

(iv)Let A be a Left ideal of N. Since N = N0, from Proposition 2.2.6 (ii) we get, NA □ A. Therefore A is an N-subgroup of N. Form Theorem 4.3.17 we get, A is an invariant N-subgroup of N.

(v)and (vi) follow from (iv).

We conclude our discussion with the following characterization of strong B1 near-rings.

Theorem 4.3.19.
Let N be a Boolean near-ring. Then N is a strong B1 near-ring if and only if Na ∩ Nb = Nab for all a,b□N.

Proof:

For the „only if" part, let a,b□N. If y□Na ∩ Nb then y = na = n'b for some n,n'□N. Now Lemma 4.3.9 demands that there exists z□N such that y2 = (na) (n'b)

= (nan')b = (zn'a)b = (zn')ab□Nab. That is y2 □ Nab. Since N is Boolean, y □ Nab. Therefore
Na ∩ Nb □ Nab -------------------------- (18)
Since N is a strong B1 near-ring, Nab = Nba. But Nba C Na and Nab C Nb. Hence

Nab □ Na∩Nb -------------------------- (19)

From 18 and 19 we get, Na∩Nb =Nab.

For the „if part", let a,b□N. Now Nab = Na∩Nb = Nb∩Na. Thus N is a strong B1 near-ring.

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