A Notes on Intuitionistic Fuzzy Differential Equations

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Absrtact - In this paper, different types of solution of the first order linear Intuitionistic Fuzzy Ordinary Differential Equations(IFODEs) are discussed by using Lagrange Multiplier Method. The initial conditions and the coefficients of differential equations which are taken as the Generalized Triangular Intuitionistic Fuzzy Numbers. Also real life problem is solved by using above method

Index Terms: Generalized Triangular Intuitionistic Fuzzy Numbers, Intuitionistic Fuzzy Differential Equations, Lagrange Multiplier Method.

I. INTRODUCTION

Fuzzy sets was introduced by Zadeh in 1965[1] its deal with the membership function. It was generalized into Intuitionistic Fuzzy Sets by Atanassov [2] in which non-membership function is also considered. Fuzzy Differential Equations (FDEs) is rapidly growing in recent years. FDEs was introduced by Kandal and Byatt in 1987 [3]. FDEs are used in science and engineering. S.P.Mondal and T.K.Roy[4] discussed the first order linear homogeneous fuzzy ordinary differential equations with generalized triangular fuzzy numbers as taken as the initial conditions. Differential equations under intuitionistic fuzzy environment was discussed by Melliani and Chadli[5]. In this paper, first order linear homogeneous intuitionistic fuzzy ordinary differential equations with application is

discussed. Generalized Triangular Intuitionistic Fuzzy Number as taken as the initial condition of the differential equations.
Solution of System of Differential Equations by Lagrange Multiplier Method:[4]
Consider the system of first order differential equations

$$\left.\begin{array}{l} \frac{dx}{dt} = a_1 x + b_1 y \\ \frac{dy}{dt} = a_2 x + b_2 y \end{array}\right\} \quad (1)$$

Multiplying the second Equation by Lagrange multiplier λ and add termwise to the first equation we get

$$\begin{aligned} \frac{d(x+\lambda y)}{dt} &= (a_1+\lambda a_2)x + (b_1+\lambda b_2)y \\ &= (a_1+\lambda a_2)(x + \frac{b_1+\lambda b_2}{a_1+\lambda a_2}y) \end{aligned} \quad (2)$$

choose the number λ so that

$$\frac{b_1+\lambda b_2}{a_1+\lambda a_2} = \lambda \quad (3)$$

Then(2) reduces to an equation linear in $x + \lambda y$

$$\frac{d(x+\lambda y)}{dt} = (a_1+\lambda a_2)(x+\lambda y)$$

which on integrating gives

$$x + \lambda y = c\exp(a_1+\lambda a_2) \quad (4)$$

If equation(3) has distinct real roots λ_1 and λ_2, then we obtain the integrals of (1)-(4).

II. SOLUTION OF FIRST ORDER LINEAR HOMOGENEOUS IFODEs

The Solution of First order homogeneous IFODEs of Type-I and Type-II are described. The intuitionistic fuzzy numbers are taken as GTIFNs.

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2.1. Solution of First Order Linear Homogeneous IFODEs of Type-I

Consider the initial value problem

$$\frac{dy}{dt} = ky \tag{5}$$

with generalized triangular intuitionistic fuzzy numbers,
$\tilde{y}(t_0) = \tilde{a_0} = ((a_1, a_2, a_3; \omega), (a'_1, a_2, a'_3; \sigma))$.
Suppose the solution of IFODE (5) is $\tilde{y}(t)$ with $(\alpha, \beta) - cut$,
$y(t, \alpha, \beta) = ([y_1(t, \alpha), y_2(t, \alpha)], [y'_1(t, \beta), y'_2(t, \beta)])$
and $(\tilde{a_0})_{(\alpha,\beta)} = \Big([a_1 + \frac{\alpha l_{a_0}}{\omega}, a_3 - \frac{\alpha r_{a_0}}{\omega}], [a'_1 - \frac{\beta l_{a_0}}{\sigma}, a'_3 + \frac{\beta r_{a_0}}{\sigma}]\Big)$
$\forall \alpha \in [0, \omega]$, $0 < \omega \leq 1$, $\forall \beta \in [0, \sigma]$, $0 < \sigma \leq 1$
To solve $k > 0$ and $k < 0$ for the given problem.
Case 2.1.1: Suppose $k > 0$, From (5),

$$\frac{dy_i(t, \alpha, \beta)}{dt} = ky_i(t, \alpha, \beta), \; i = 1, 2 \tag{6}$$

The result of (6) is given by,
$([y_1(t, \alpha) = (a_1 + \frac{\alpha l_{a_0}}{\omega})\, e^{k(t-t_0)}, \; y_2(t, \alpha) = (a_3 - \frac{\alpha r_{a_0}}{\omega})\, e^{k(t-t_0)}]$,
$[y'_1(t, \beta) = (a'_1 - \frac{\beta l_{a_0}}{\sigma})\, e^{k(t-t_0)}, \; y'_2(t, \beta) = (a'_3 + \frac{\beta r_{a_0}}{\sigma})\, e^{k(t-t_0)}])$
Suppose,
$\frac{\partial}{\partial \alpha}[y_1(t, \alpha)] > 0, \; \frac{\partial}{\partial \alpha}[y_2(t, \alpha)] < 0, \; \frac{\partial}{\partial \beta}[y'_1(t, \beta)] < 0, \; \frac{\partial}{\partial \beta}[y'_2(t, \beta)] > 0$
and $y_1(t, \omega) \leq y_2(t, \omega)$, $y'_1(t, \sigma) \leq y'_2(t, \sigma)$.
Then the solution is strong.
Case 2.1.2: Suppose $k < 0$, Let $k = -m$ where m is a positive real number. The IFODE(6) follows,

$$\frac{dy_1(t, \alpha)}{dt} = -my_2(t, \alpha), \; \frac{dy_2(t, \alpha)}{dt} = -my_1(t, \alpha) \tag{7}$$

$$\frac{dy'_1(t, \beta)}{dt} = -my'_2(t, \beta), \; \frac{dy'_2(t, \beta)}{dt} = -my'_1(t, \beta) \tag{8}$$

To find the membership function of the solutuion:
From equation (7),

$$\frac{d}{dt}[y_1(t, \alpha) + \lambda y_2(t, \alpha)] = -\lambda m[y_1(t, \alpha) + \frac{1}{\lambda} y_2(t, \alpha)]$$

Let $\frac{1}{\lambda} = \lambda$ and $y_1(t, \alpha) + \lambda y_2(t, \alpha) = z$
Hence the solution is, $z = Ce^{-\lambda mt}$ and $\lambda = \pm 1$

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Then,

$$y_1(t,\alpha) + y_2(t,\alpha) = C_1 e^{-mt}$$
$$y_1(t,\alpha) - y_2(t,\alpha) = C_2 e^{mt}$$

Applying the initial conditions and solve we get,

$$y_1(t,\alpha) = \frac{1}{2}\left\{a_1 + a_3 + \frac{\alpha}{\omega}(l_{a_0} - r_{a_0})\right\} e^{-m(t-t_0)} + \frac{1}{2}(\frac{\alpha}{\omega} - 1)(l_{a_0} + r_{a_0})e^{m(t-t_0)}$$
$$y_2(t,\alpha) = \frac{1}{2}\left\{a_1 + a_3 + \frac{\alpha}{\omega}(l_{a_0} - r_{a_0})\right\} e^{-m(t-t_0)} - \frac{1}{2}(\frac{\alpha}{\omega} - 1)(l_{a_0} + r_{a_0})e^{m(t-t_0)}$$

To find the non - membership function of the solution:
From (8) we get,

$$\frac{d}{dt}[y_1'(t,\beta) + \lambda y_2'(t,\beta)] = -\lambda m[y_1'(t,\beta) + \frac{1}{\lambda} y_2'(t,\beta)]$$

Let $\frac{1}{\lambda} = \lambda$ and $y_1'(t,\beta) + \lambda y_2'(t,\beta) = z_1$
Then, $z_1 = Ce^{-\lambda mt}$ and $\lambda = \pm 1$
Hence,

$$y_1'(t,\beta) + y_2'(t,\beta) = C_3 e^{-mt}$$

$$y_1'(t,\beta) - y_2'(t,\beta) = C_4 e^{mt}$$

Applying the initial conditions and solve,

$$y_1'(t,\beta) = \frac{1}{2}\left\{a_1' + a_3' - \frac{\beta}{\sigma}(l_{a_0} - r_{a_0})\right\} e^{-m(t-t_0)} - \frac{1}{2}(\frac{\beta}{\sigma} - 1)(l_{a_0} + r_{a_0})e^{m(t-t_0)}$$
$$y_2'(t,\beta) = \frac{1}{2}\left\{a_1' + a_3' - \frac{\beta}{\sigma}(l_{a_0} - r_{a_0})\right\} e^{-m(t-t_0)} + \frac{1}{2}(\frac{\beta}{\sigma} - 1)(l_{a_0} + r_{a_0})e^{m(t-t_0)}$$

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Suppose,
$\frac{\partial}{\partial\alpha}[y_1(t,\alpha)] > 0,\ \frac{\partial}{\partial\alpha}[y_2(t,\alpha)] < 0,\ \frac{\partial}{\partial\beta}[y_1'(t,\beta)] < 0,\ \frac{\partial}{\partial\beta}[y_2'(t,\beta)] > 0$
and $y_1(t,\omega) \leq y_2(t,\omega),\ y_1'(t,\sigma) \leq y_2'(t,\sigma)$.
Hence the solution is strong.
2.2. Solution of First Order Linear Homogeneous IFODEs of Type-II
Consider the initial value problem

$$\frac{dy}{dt} = \tilde{k}y \tag{9}$$

with the initial condition $y(t_0) = \gamma$,
where $\tilde{k} = ((b_1, b_2, b_3; \lambda), (b_1', b_2, b_3'; \delta))$.
Suppose the solution of IFODE (9) is $\tilde{y}(t)$.
Let $y(t,\alpha,\beta) = ([y_1(t,\alpha), y_2(t,\alpha)], [y_1'(t,\beta), y_2'(t,\beta)])$ be the $(\alpha,\beta) - cut$ of the solution. The $(\alpha,\beta) - cut$ of $\tilde{k}$ be
$(\tilde{k})_{(\alpha,\beta)} = ([b_1 + \frac{\alpha l_k}{\lambda}, b_3 - \frac{\alpha r_k}{\lambda}], [b_1' - \frac{\beta l_k}{\delta}, b_3' + \frac{\beta r_k}{\delta}])$
$\forall\alpha \in [0,\lambda],\ 0 < \lambda \leq 1,\ \forall\beta \in [0,\delta],\ 0 < \delta \leq 1$
To solve the problem for $\tilde{k} > 0$ and $\tilde{k} < 0$.
Case 2.2.1: Suppose $\tilde{k} > 0$
From (9),

$$\frac{dy_i(t,\alpha,\beta)}{dt} = k_i(\alpha,\beta)y_i(t,\alpha,\beta),\ i = 1,2 \tag{10}$$

The solution of (10) is,
$([y_1(t,\alpha) = \gamma e^{(b_1+\frac{\alpha l_k}{\lambda})(t-t_0)},\ y_2(t,\alpha) = \gamma e^{(b_3-\frac{\alpha r_k}{\lambda})(t-t_0)}],$
$[y_1'(t,\beta) = \gamma e^{(b_1'-\frac{\beta l_k}{\delta})(t-t_0)}, y_2'(t,\beta) = \gamma e^{(b_3'+\frac{\beta r_k}{\delta})(t-t_0)}])$
Suppose,
$\frac{\partial}{\partial\alpha}[y_1(t,\alpha)] > 0,\ \frac{\partial}{\partial\alpha}[y_2(t,\alpha)] < 0,\ \frac{\partial}{\partial\beta}[y_1'(t,\beta)] < 0,\ \frac{\partial}{\partial\beta}[y_2'(t,\beta)] > 0$
and $y_1(t,\lambda) \leq y_2(t,\lambda),\ y_1'(t,\delta) \leq y_2'(t,\delta)$.
Then the solution is strong.
Case 2.2.2: Suppose $\tilde{k} < 0$,
Let $\tilde{k} = -m$, $m = ((b_1, b_2, b_3; \lambda), (b_1', b_2, b_3'; \delta))$ is a positive GTIFNs.
$(m)_{(\alpha,\beta)} = \{[m_1(\alpha), m_2(\alpha)]; [m_1'(\beta), m_2'(\beta)]\}$
$= ([b_1 + \frac{\alpha l_m}{\lambda}, b_3 - \frac{\alpha r_m}{\lambda}], [b_1' - \frac{\beta l_m}{\delta}, b_3' + \frac{\beta r_m}{\delta}]),$
$\forall\alpha \in [0,\lambda],\ 0 < \lambda \leq 1,\ \forall\beta \in [0,\delta],\ 0 < \delta \leq 1$

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From (10),

$$\frac{dy_1(t,\alpha)}{dt} = -m_2(\alpha)y_2(t,\alpha), \quad \frac{dy_2(t,\alpha)}{dt} = -m_1(\alpha)y_1(t,\alpha) \qquad (11)$$

$$\frac{dy_1'(t,\beta)}{dt} = -m_2(\beta)y_2'(t,\beta), \quad \frac{dy_2'(t,\beta)}{dt} = -m_1(\beta)y_1'(t,\beta) \qquad (12)$$

To find the membership function of the solution : From(11)

$$\frac{d}{dt}[y_1(t,\alpha) + \lambda y_2(t,\alpha)] = -\lambda m[y_1(t,\alpha) + \frac{m_2(\alpha)}{\lambda m_1(\alpha)} y_2(t,\alpha)]$$

Take $\lambda = \frac{m_2(\alpha)}{\lambda m_1(\alpha)}$ and choose $y_1(t,\alpha) + \lambda y_2(t,\alpha) = z$
The result is,
$z = Ce^{-\lambda m_1 t}$ and $\lambda = \pm\sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}}$
Then,

$$y_1(t,\alpha) + \sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}} y_2(t,\alpha) = C_1 e^{-\sqrt{m_1(\alpha)m_2(\alpha)}t}$$

$$y_1(t,\alpha) - \sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}} y_2(t,\alpha) = C_2 e^{\sqrt{m_1(\alpha)m_2(\alpha)}t}$$

To solve and applying the initial conditions ,
$y_1(t,\alpha) = \frac{\gamma}{2}$
$\left\{\left(1 - \sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}}\right) e^{\sqrt{m_1(\alpha)m_2(\alpha)}(t-t_0)} + \left(1 + \sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}}\right) e^{-\sqrt{m_1(\alpha)m_2(\alpha)}(t-t_0)}\right\}$
$y_2(t,\alpha) = \frac{\gamma}{2}\sqrt{\frac{m_1(\alpha)}{m_2(\alpha)}}$
$\left\{-\left(1 - \sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}}\right) e^{\sqrt{m_1(\alpha)m_2(\alpha)}(t-t_0)} + \left(1 + \sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}}\right) e^{-\sqrt{m_1(\alpha)m_2(\alpha)}(t-t_0)}\right\}$
To find the non - membership function of the solution: From(12)

$$\frac{d}{dt}[y_1'(t,\beta) + y_2'(t,\beta)] = -\lambda m_1'[y_1'(t,\beta) + \frac{m_2'(\beta)}{\lambda m_1'(\beta)} y_2'(t,\beta)]$$

Let $\lambda = \frac{m_2'(\beta)}{\lambda m_1'(\beta)}$ and choose $y_1'(t,\beta) + \lambda y_2'(t,\beta) = z_1$
The result is ,
$z_1 = Ce^{-\lambda m_1' t}$ and $\lambda = \pm\sqrt{\frac{m_2'(\beta)}{m_1'(\beta)}}$
Then

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$$y_1'(t,\beta)+\sqrt{\frac{m_2'(\beta)}{m_1'(\beta)}}y_2'(t,\beta)=C_3e^{-\sqrt{m_1'(\beta)m_2'(\beta)}t}$$
$$y_1'(t,\beta)-\sqrt{\frac{m_2'(\beta)}{m_1'(\beta)}}y_2'(t,\beta)=C_4e^{\sqrt{m_1'(\beta)m_2'(\beta)}t}$$

To solve and applying the initial conditions,
$y_1'(t,\beta)=\frac{\gamma}{2}$
$\left\{\left(1-\sqrt{\frac{m_2'(\beta)}{m_1'(\beta)}}\right)e^{\sqrt{m_1'(\beta)m_2'(\beta)}(t-t_0)}+\left(1+\sqrt{\frac{m_2'(\beta)}{m_1'(\beta)}}\right)e^{-\sqrt{m_1'(\beta)m_2'(\beta)}(t-t_0)}\right\}$
$y_2'(t,\beta)=\frac{\gamma}{2}\sqrt{\frac{m_1'(\beta)}{m_2'(\beta)}}$
$\left\{-\left(1-\sqrt{\frac{m_2'(\beta)}{m_1'(\beta)}}\right)e^{\sqrt{m_1'(\beta)m_2'(\beta)}(t-t_0)}+\left(1+\sqrt{\frac{m_2'(\beta)}{m_1'(\beta)}}\right)e^{-\sqrt{m_1'(\beta)m_2'(\beta)}(t-t_0)}\right\}$
Suppose,
$\frac{\partial}{\partial\alpha}[y_1(t,\alpha)]>0,\ \frac{\partial}{\partial\alpha}[y_2(t,\alpha)]<0,\ \frac{\partial}{\partial\beta}[y_1'(t,\beta)]<0,\ \frac{\partial}{\partial\beta}[y_2'(t,\beta)]>0$
and $y_1(t,\lambda)\leq y_2(t,\lambda),\ y_1'(t,\delta)\leq y_2'(t,\delta)$.
Hence the solution is strong.

III. APPLICATION PROBLEM

The Balance A(t) of bank account is develop under the process by $\frac{dA}{dt}=iA$,where i is the constant proportionality of the annual interest rate. There initially $A(t)=A_0$ balance, solve the above problem in intuitionistic fuzzy environment when,

1. $A_0=((85,100,125;0.5),(98,100,105;0.5))$ and $i=4\%$
2. $A_0=100$ and $i=((3,4,7;0.3),(2,4,5;0.3))\%$

Solution:
Type 1: The solution is,
$A_1(t,\alpha)=(85+30\alpha)e^{0.04t},\ \ A_2(t,\alpha)=(125-50\alpha)e^{0.04t}$
$A_1'(t,\beta)=(98-4\beta)e^{0.04t},\ \ A_2'(t,\beta)=(105+10\beta)e^{0.04t}$

Then,

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Table 1

For t = 4		
α	$A_1(t,\alpha)$	$A_2(t,\alpha)$
0	99.7475	146.687
0.1	103.268	140.820
0.2	106.788	134.952
0.3	110.309	129.085
0.4	113.829	123.217
0.5	117.350	117.350

For t = 4		
β	$A'_1(t,\beta)$	$A'_2(t,\beta)$
0	115.003	123.217
0.1	114.533	124.391
0.25	113.829	126.151
0.3	113.594	126.738
0.47	112.796	128.732
0.5	112.656	129.085

In Table 1, the values of t, $A_1(t,\alpha)$ and $A'_2(t,\beta)$ are increasing functions, $A_2(t,\alpha)$ and $A'_1(t,\beta)$ are decreasing functions. Here, $A_1(t,0.5) = A_2(t,0.5)$ and $A'_1(t,0) < A'_2(t,0)$. It satisfies the conditions so we get a strong solution.

Type 2: The solution is,
$A_1(t,\alpha) = 100\ e^{\frac{3+3.33\alpha}{100}5}$, $A_2(t,\alpha) = 100\ e^{\frac{7-10\alpha}{100}5}$
$A'_1(t,\beta) = 100\ e^{\frac{2-6.66\beta}{100}5}$, $A'_2(t,\beta) = 100\ e^{\frac{5+3.33\beta}{100}5}$
Now,

Table 2

For t = 5		
α	$A_1(t,\alpha)$	$A_2(t,\alpha)$
0	116.183	141.906
0.15	119.121	131.653
0.28	121.728	123.367
0.3	122.134	122.140

For t = 5		
β	$A'_1(t,\beta)$	$A'_2(t,\beta)$
0	110.517	128.402
0.1	106.897	130.558
0.2	108.697	132.750
0.3	100.010	134.979

$A_1(t,\alpha)$ and $A'_2(t,\beta)$ are increasing functions for various values of t, while $A_2(t,\alpha)$ and $A'_1(t,\beta)$ are decreasing functions. Here $A_1(t,0.3) < A_2(t,0.3)$ and $A'_1(t,0) < A'_2(t,0)$. It satisfies the conditions, therefore the solution is strong.

IV. CONCLUSION

Generalized Triangular intuitionistic fuzzy number is taken in two ways: (i) initial value, (ii)coefficients. Solution of first order linear homogeneous intuitionistic fuzzy ordinary differential equation is obtained. The bank account problem is solved.

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