ANTI PERIODIC BOUNDARY VALUE PROBLEM FOR FRACTIONAL DIFFERENTIAL EQUATIONS

S.Padmavathi[1], C.Ugambika[2]
Department of Mathematics
Vellalar College for Women(Autonomous), Erode
India

Abstract - In this paper, we study the existence of the positive solution of the nonlinear antiperiodic boundary value problem

$$\mathbf{D^{\alpha}_{0+}x(t) = f(t, x(t)), 0 < t < 1}$$
$$\mathbf{x(0) + x(1) = 0, x'(0) + x'(1) = 0}$$

where $1 < \alpha \leq 2$ is a real number and $\mathbf{D^{\alpha}_{0+}}$ is the Caputo's fractional derivative and $\mathbf{f : [0, 1] \times [0, \infty) \to [0, \infty)}$ is continuous. We study the conditions for existence of positive solutions by means of fixed point theorem on cones.

Index Terms: Antiperiodic BVP, Caputo derivative, Cone, Fixed point theorem, Green's function, Positive Solution.

I. INTRODUCTION

The Fractional differential equations involves the derivatives of fractional order, which plays a significant role in aerodynamics, physics, chemistry, biosciences, etc., especially in the fields of engineering and information technology.

Fractional derivative equip with mostly by two operators say Riemann Liouville and Caputo operators. Meanwhile Caputo operator efficiently involves in solving both initial value problems as well as boundary value problems.

In [8], the conditions for existence and multiplicity of positive solutions of nonlinear fractional differential equations with the boundary value problem involving Caputo's derivative.

$$D^{\alpha}_{0+}x(t) = f(t, x(t)),\ t \in (0, 1)$$

1

$$x(0) + x'(0) = 0, x(1) + x'(1) = 0$$

where $1 < \alpha \leq 2$ is a real number and D^{α}_{0+} is the Caputo's fractional derivative and $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous.

Motivated by the above work, it is proposed to investigate the existence of positive solutions for antiperiodic nonlinear boundary value problem of fractional order.

$$\begin{aligned} D^{\alpha}_{0+}x(t) &= f(t, x(t)),\ t \in (0, 1) \\ x(0) + x(1) &= 0, x'(0) + x'(1) = 0 \end{aligned} \tag{1}$$

where $1 < \alpha \leq 2$ is a real number and $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous. The existence of positive solutions is obtained by means of a fixed point theorem on cones.

II. PRELIMINARIES

In this section, we give some important definitions, lemmas and some preliminaries which are used throughout by this paper.

Definition 2.1. The Riemann - Liouville fractional integral of order α can be written as

$$I^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s)ds, t > 0$$

where $\alpha > 0$.

Definition 2.2.[7] For a function $f(x)$ given in the interval $[0, \infty)$, the expression

$$D^{\alpha}_{0+}f(x) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(t)}{(x - t)^{\alpha-n+1}} dt$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, is called the Riemann-Liouville fractional derivative of order α.

Lemma 2.1.[8] Let $\alpha > 0$, then the differential equation

$$D^{\alpha}_{0+}x(t) = 0$$

has solutions $x(t) = c_0 + c_1t + c_2t^2 + ... + c_nt^{n-1}$, $c_i \in \mathcal{R}$, $i = 0, 1, ..., n, n = [\alpha] + 1$.

Lemma 2.2.[8] Let $\alpha > 0$, then

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) + c_0 + c_1t + c_2t^2 + ... + c_nt^{n-1}$$

2

for some $c_i \in \mathcal{R}$, $i = 0, 1, ..., n, n = [\alpha] + 1$.

Lemma 2.3.[4] Let X be a Banach space, and let $P \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let $S : P \longrightarrow P$ be a completely continuous operator such that, either

(1) $\|Sw\| \le \|w\|, w \in P \cap \partial\Omega_1, \|Sw\| \ge \|w\|, w \in P \cap \partial\Omega_2$, or

(2) $\|Sw\| \ge \|w\|, w \in P \cap \partial\Omega_1, \|Sw\| \le \|w\|, w \in P \cap \partial\Omega_2$.

Then S has fixed point in $P \cap \overline{\Omega_2} | \Omega_1$.

Definition 2.3.[8] A map δ is said to be a nonnegative continuous concave functional on K if $\delta : K \longrightarrow [0, +\infty)$ is continuous and

$$\delta(tx + (1-t)y) \ge t\delta(x) + (1-t)\delta(y)$$

for all $x, y \in K$ and $0 \le t \le 1$. And let

$$K(\delta, a, b) = \{u \in K | a \le \delta(u), \|u\| \le b\}$$

III. MAIN RESULTS

Lemma 3.1. Let $w(t) \in C[0,1]$ be a given function, then the boundary value problem,

$$\begin{gathered} D^{\alpha}_{0+}x(t) = f(t, x(t)), 0 < t < 1 \\ x(0) + x(1) = 0, x'(0) + x'(1) = 0 \end{gathered} \tag{2}$$

has a unique solution

$$x(t) = \int_0^1 G(t,s)h(s)ds \tag{3}$$

where

$$G(t,s) = \begin{cases} \frac{2(t-s)^{\alpha-1} - (1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2} - 2t(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}, s \le t \\ \\ \frac{(1-s)^{\alpha-2} - 2t(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)} - \frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)}, t \le s \end{cases} \tag{4}$$

Here G(t, s) is called the Green's function of the given boundary value problem.
Proof. Let

$$D^{\alpha}_{0+}x(t) = w(t) \tag{5}$$

Since

$$I^{\alpha}_{0+}D^{\alpha}_{0+}x(t) = x(t) + c_1 + c_2t + ...$$

3

$$I^{\alpha}_{0+}x(t) = x(t) + c_1 + c_2 t + ...$$
$$x(t) = I^{\alpha}_{0+}w(t) - c_1 - c_2 t$$

From the definition,

$$I^{\alpha}_{0+}w(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}w(s)ds$$
$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}w(s)ds - c_1 - c_2 t$$

for some constants $c_1, c_2 \in \mathcal{R}$. By using the relations $D^{\alpha}_{0+}I^{\alpha}_{0+}x(t) = x(t)$ and $I^{\alpha}_{0+}I^{\beta}_{0+}x(t) = I^{\alpha+\beta}_{0+}$, where $\alpha, \beta > 0$; $x \in L(0,1)$[7], we have

$$x'(t) = D^{1}_{0+}I^{\alpha}_{0+}w(t) - c_2$$
$$= I^{\alpha-1}_{0+}w(t) - c_2$$
$$x'(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}w(s)ds - c_2$$

As applying the boundary conditions we have,

$$x(0) = -c_1$$
$$x'(0) = -c_2$$
$$x(1) = I^{\alpha}_{0+}c_0 - c_1 - c_2$$
$$x'(1) = I^{\alpha-1}_{0+}c_0 - c_2$$

Since,

$$x(0) + x(1) = 0$$
$$-c_1 + I^{\alpha}_{0+}c_0 - c_1 - c_2 = 0$$
$$I^{\alpha}_{0+}c_0 - 2c_1 = c_2 \tag{6}$$

and also $x'(0) + x'(1) = 0$

$$I^{\alpha-1}_{0+}c_0 - 2c_2 = 0 \tag{7}$$

substituting (6) in (7) we get,

$$I^{\alpha-1}_{0+}c_0 - 2I^{\alpha}_{0+}c_0 + 4c_1 = 0$$
$$c_1 = \frac{1}{4}\left[2I^{\alpha}_{0+}c_0 - I^{\alpha-1}_{0+}c_0\right]$$

4

Therefore the unique solution is,

$$
\begin{aligned}
x(t) &= I_{0+}^{\alpha} w(t) - \frac{1}{4}\left[2I_{0+}^{\alpha}c_0 - I_{0+}^{\alpha-1}c_0\right] \\
&- t\left[I_{0+}^{\alpha}c_0 - 2\left[\frac{1}{4}\left(2I_{0+}^{\alpha}c_0 - I_{0+}^{\alpha-1}c_0\right)\right]\right] \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}w(s)ds - \frac{1}{2\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}w(s)ds \\
&+ \frac{1}{4\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}w(s)ds - \frac{t}{2\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}w(s)ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}w(s)ds - \frac{1}{2\Gamma(\alpha)}\int_0^t (1-s)^{\alpha-1}w(s)ds \\
&- \frac{1}{2\Gamma(\alpha)}\int_t^1 (1-s)^{\alpha-1}w(s)ds + \frac{1}{4\Gamma(\alpha-1)}\int_0^t (1-s)^{\alpha-2}w(s)ds \\
&+ \frac{1}{4\Gamma(\alpha-1)}\int_t^1 (1-s)^{\alpha-2}w(s)ds - \frac{t}{2\Gamma(\alpha-1)}\int_0^t (1-s)^{\alpha-2}w(s)ds \\
&- \frac{t}{2\Gamma(\alpha-1)}\int_t^1 (1-s)^{\alpha-2}w(s)ds \\
&= \int_0^t \left[\frac{2(t-s)^{\alpha-1} - (1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2} - 2t(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}\right] w(s)ds \\
&+ \int_t^1 \left[\frac{(1-s)^{\alpha-2} - 2t(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)} - \frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)}\right] w(s)ds \\
&= \int_0^1 G(t,s)w(s)ds
\end{aligned}
$$

which completes the proof.

Lemma 3.2. Let w(t) $\in C[0,1]$ be a given function, then function G(t, s) defined by (4) has the following properties:

($R1$) G(t, s) $\in C\left([0,1]\times[0,1)\right)$ and $G(t,s) > 0$ for $t, s \in (0,1)$

($R2$) There exists a function $\chi \in C(0,1)$ such that

$$\min_{1/4\le t\le 3/4} G(t,s) \ge |\chi(s)|\, H(s), s \in (0,1)$$

$$\max_{0\le t\le 1} G(t,s) \le H(s), \tag{8}$$

where

$$H(s) = \frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}, s \in [0,1) \tag{9}$$

5

Proof. From G(t, s), it is obvious that $G(t,s) \in C([0,1] \times [0,1))$ and $G(t,s) \geq 0$ for $s, t \in (0,1)$. Next we shall prove $(R2)$. From the definition of G(t, s), we know that for a given $s \in (0,1)$, G(t, s) is decreasing function with respect to t for $t \leq s$,

$$g_1(t,s) = \frac{2(t-s)^{\alpha-1} - (1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2} - 2t(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}, s \leq t$$

$$g_2(t,s) = \frac{(1-s)^{\alpha-2} - 2t(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)} - \frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)}, t \leq s$$

That is, $g_1(t,s)$ is a continuous function for $1/4 \leq t \leq 3/4$, and $g_2(t,s)$ is decreasing with respect to t. Hence, we have

$$g_1(t,s) \geq -\frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} - \frac{(1-s)^{\alpha-2}}{8\Gamma(\alpha-1)}, \text{ for } 1/4 \leq t \leq 3/4$$

$$\max_{0 \leq t \leq 1} g_1(t,s) \leq \frac{2(1-s)^{\alpha-1} - (1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)} \leq \frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}$$

$$\min_{1/4 \leq t \leq 3/4} g_2(t,s) = g_2(3/4, s) = -\frac{(1-s)^{\alpha-2}}{8\Gamma(\alpha-1)} - \frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)}$$

$$\begin{aligned}\max_{0 \leq t \leq 1} g_2(t,s) = g_2(0,s) &= \frac{(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)} - \frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} \\ &< \frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}\end{aligned}$$

Thus we have,

$$\min_{1/4 \leq t \leq 3/4} G(t,s) \geq h(s) = -\frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} - \frac{(1-s)^{\alpha-2}}{8\Gamma(\alpha-1)}, s \in [0,1) \quad (10)$$

$$\max_{0 \leq t \leq 1} G(t,s) \leq H(s) = \frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}, s \in [0,1) \quad (11)$$

Let

$$
\begin{aligned}
|\chi| &= \left|\frac{h(s)}{H(s)}\right| = \frac{|h(s)|}{|H(s)|} \\
&= \left|\frac{1}{2}\left(\frac{-4(1-s)^{\alpha-1}-(\alpha-1)(1-s)^{\alpha-2}}{2(1-s)^{\alpha-1}+(\alpha-1)(1-s)^{\alpha-2}}\right)\right| \\
&= \frac{1}{2}\left(\frac{4(1-s)^{\alpha-1}+(\alpha-1)(1-s)^{\alpha-2}}{2(1-s)^{\alpha-1}+(\alpha-1)(1-s)^{\alpha-2}}\right), s \in (0,1) \qquad (12)
\end{aligned}
$$

Hence $|\chi(s)| \in C((0,1),(0,+\infty))$.
This completes the proof.

Remark 3.1. From the above lemma we assume that, $|\chi(s)| \geq \frac{1}{4}$.

$M = C[0,1]$ have the ordering $x \leq v$ if $x(t) \leq v(t) \forall$ t $\in [0,1]$ and the maximum norm $\|x\| = \max_{0\leq t\leq 1}|x(t)|$. The cone $K \subset M$ is defined by

$$K = \left\{x \in M |\ x(t) \geq 0, \min_{1/4\leq t\leq 3/4} \geq \frac{1}{4}\|x\|\right\}$$

and the non-negative continuous concave functional Ψ on the cone K is given by

$$\Psi(x) = \min_{1/4\leq t\leq 3/4}|x(t)|$$

Lemma 3.3. Assume that $f(t,x)$ is continuous on $[0,1]\times[0,\infty)$. A function $u \in K$ is a solution of the given boundary value problem iff it is a solution of the integral equation(3).
Proof. Let $x \in K$ be the solution of the given boundary value problem. Applying the operator I^{α}_{0+} to both sides of (1), Then we have

$$x(t) = c_1 + c_2 t + I^{\alpha}_{0+} f(t, x(t))$$

for some constants $c_1, c_2 \in \mathcal{R}$. By using the boundary conditions and by the same method obtaining the Green's function of the problem (1), calculate the constants c_1 and c_2, so

$$x(t) = \int_0^1 G(t,s) f(s, x(s)) ds$$

Since from the above lemma we get that $\int_0^1 G(t,s)f(s,x(s))ds \in K$. Hence u is

7

also a solution of the integral equation $\int_0^1 G(t,s)w(s)ds$. By applying the Caputo's fractional operator to both the sides of the integral equation (3) and denoting the right hand side of the integral equation by $Z(t)$, then by the definition of function G(t,s),

$$\begin{aligned} Z(t) &= \int_0^1 G(t,s)f(s,x(s))ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}w(s)ds - \frac{1}{2\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}w(s)ds \\ &+ \frac{1}{4\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}w(s)ds - \frac{t}{2\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}w(s)ds \end{aligned}$$

Therefore,

$$\begin{aligned} Z'(t) &= \frac{d}{dt}I_{0+}^{\alpha}f(t,x(t)) - \frac{I_{0+}^{\alpha-1}f(1,x(1))}{2} \\ &= D_{0+}^{1}I_{0+}^{\alpha}f(t,x(t)) - \frac{I_{0+}^{\alpha-1}f(1,x(1))}{2} \\ &= D_{0+}^{1}I_{0+}^{1}I_{0+}^{\alpha-1}f(t,x(t)) - \frac{I_{0+}^{\alpha-1}f(1,x(1))}{2} \\ &= I_{0+}^{\alpha}f(t,x(t)) - \frac{I_{0+}^{\alpha-1}f(1,x(1))}{2} \end{aligned}$$

and

$$\begin{aligned} Z''(t) &= \frac{d}{dt}I_{0+}^{\alpha-1}f(t,x(t)) \\ &= D_{0+}^{1}\frac{1}{D_{0+}^{\alpha-1}}f(t,x(t)) \\ &= D_{0+}^{1}D_{0+}^{-\alpha+1}f(t,x(t)) \\ Z''(t) &= D_{0+}^{-\alpha+2}f(t,x(t)) \end{aligned}$$

and

$$\begin{aligned} &Z''(t)I_{0+}^{2-\alpha} = f(t,x(t)) \\ &D_{0+}^{\alpha}Z(t) = I_{0+}^{2-\alpha}Z''(t) = D_{0+}^{2-\alpha}I_{0+}^{2-\alpha}f(t,x(t)) = f(t,x(t)) \end{aligned}$$

here, the relations $I_{0+}^{s}I_{0+}^{t}g(t) = I_{0+}^{s+t}g(t)$, $D_{0+}^{s}I_{0+}^{s}g(t) = g(t)$, $s > 0, t > 0$, $g \in L(0,1)$ and $I_{0+}^{s}D_{0+}^{s}g(t) = g(t)$,$s > 0$, $g \in C[0,1]$ and used, where D_{0+}^{s} is a Riemann - Liouville fractional derivative. That is, $D_{0+}^{\alpha}x(t) = f(t,x(t))$. Now,

$$x(0) = \int_0^1 \frac{(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)} - \frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)}f(s,x(s))ds$$

8

$$x(1) = \int_0^1 \frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)} - \frac{(1-s)^{\alpha-1}}{4\Gamma(\alpha-1)} f(s, x(s))ds$$
$$x'(0) = -\frac{1}{2}\int_0^1 (1-s)^{\alpha-2} f(s, x(s))ds$$
$$x'(1) = \frac{1}{2}\int_0^1 (1-s)^{\alpha-2} f(s, x(s))ds$$

Therefore we obtain , $x(0) + x(1) = 0, x'(0) + x'(1) = 0$, which implies $x \in K$ is a solution (1).

Lemma 3.4. Assume that $f(t, x)$ is continuous on $[0, 1] \times [0, \infty)$ and the operator $Q : K \to M$ is defined by

$$Qx(t) = \int_0^1 G(t, s)f(s, x(s))ds$$

Then $Q : K \to K$ is completely continuous.
Proof. From the expression of Green's function it is clear that, $Qx(t) \geq 0$, $t \in [0, 1]$, $Qx(t)$ is continuous for $x \in K$. Then by Lemma (3.1) and remark (3.1), we have

$$\min_{1/4 \leq t \leq 3/4} Qx(t) = \min_{1/4 \leq t \leq 3/4} \int_0^1 G(t, s)f(s, x(s))ds \geq \frac{1}{4}\int_0^1 H(s)f(s, x(s))ds$$

and

$$\|Qx\| = \max_{0 \leq t \leq 1} |Qx(t)| \leq \int_0^1 H(s)f(s, x(s))ds.$$

Thus, we get

$$\min_{1/4 \leq t \leq 3/4} Qx(t) \geq \frac{1}{4}\|Qx\|$$

which implies $Q : K \to K$.

Let $R \subset K$ be bounded. That is there exists a positive constant $L > 0$ such that $\|x\| \leq L$, for all $x \in R$. Let $H = \max_{0 \leq t \leq 1, 0 \leq u \leq L} |f(t, x)| + 1$, then for $x \in R$, from lemma(3.1) we have,

$$|Qx(t)| \leq \int_0^1 |G(t, s)f(t, x(s))| \, ds \leq H\int_0^1 H(s)ds$$

Hence, $Q(R)$ is bounded. For all $\epsilon > 0$, each $x \in R, t_1, t_2 \in [0,1], t_1 < t_2$, let

$$\delta = min\left\{\frac{1}{2}, \frac{2\Gamma(\alpha)\epsilon}{6H}, \frac{\Gamma(1+\alpha)\epsilon}{8H}\right\}$$

Now we shall prove that $|Qx(t_2) - Qx(t_1)| < \epsilon$, when $t_2 - t_1 < \delta$.
Consider

$$= \left|\int_0^1 G(t,s)f(s,x(s))ds - \int_0^1 G(t,s)f(s,x(s))ds\right|$$
$$\leq \int_0^{t_1} |(G(t_2,s) - G(t_1,s))f(s,x(s)|\, ds + \int_{t_2}^1 |(G(t_2,s) - G(t_1,s))f(s,x(s)|\, ds$$
$$+ \int_{t_1}^{t_2} |(G(t_2,s) - G(t_1,s))f(s,x(s)|\, ds$$
$$\leq H(\int_0^{t_1} |(G(t_2,s) - G(t_1,s))|\, ds + \int_{t_2}^1 |G(t_2,s) - G(t_1,s))|\, ds$$
$$+ \int_{t_1}^{t_2} |G(t_2,s) - G(t_1,s))|\, ds)$$
$$= H(\int_0^{t_1} \frac{2[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] - (1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{(2t_2-2t_1)(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)}$$
$$+ \int_{t_2}^1 \frac{(2t_2-2t_1)(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)} - \frac{(1-s)^{\alpha-1}}{2\Gamma(\alpha)}$$
$$+ \int_{t_1}^{t_2} \frac{2(t_2-s)^{\alpha-1} - (1-s)^{\alpha-1}}{2\Gamma(\alpha)} + \frac{(2t_2-2t_1)(1-s)^{\alpha-2}}{4\Gamma(\alpha-1)})ds$$
$$= H(\int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\delta(1-s)^{\alpha-2}}{2\Gamma(\alpha-1)} + \int_{t_2}^1 \frac{\delta(1-s)^{\alpha-2}}{2\Gamma(\alpha-1)}$$
$$+ \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\delta(1-s)^{\alpha-2}}{2\Gamma(\alpha-1)})ds$$
$$\leq H\left(\frac{t_2^\alpha - t_1^\alpha}{\Gamma(\alpha+1)} + \frac{\delta}{2\Gamma(\alpha)} + \frac{\delta}{2\Gamma(\alpha)} + \frac{2\delta^\alpha}{\Gamma(\alpha+1)} + \frac{\delta}{2\Gamma(\alpha)}\right)$$
$$= H\left(\frac{3\delta}{2\Gamma(\alpha)} + \frac{2\delta^\alpha + (t_2^\alpha - t_1^\alpha)}{\Gamma(\alpha+1)}\right) < H\left(\frac{3\delta}{2\Gamma(\alpha)} + \frac{2\delta + (t_2^\alpha - t_1^\alpha)}{\Gamma(\alpha+1)}\right)$$

In order to estimate $t_2^\alpha - t_1^\alpha$, for $\delta \leq t_1 < t_2 \leq 1$, by means of mean value theorem we have,

$$t_2^\alpha - t_1^\alpha \leq \alpha(t_2 - t_1) < \alpha\delta \leq 2\delta$$

10

for $0 \leq t_1 < \delta$, $t_2 < 2\delta$, we have

$$t_2^\alpha - t_1^\alpha \leq t_2^\alpha < (2\delta)^\alpha \leq 2\delta$$

for $0 \leq t_1 < t_2 \leq \delta$, we have

$$t_2^\alpha - t_1^\alpha \leq t_2^\alpha < \delta^\alpha < 2\delta$$

Hence, we get

$$\begin{aligned} |Qx(t_2) - Qx(t_1)| &< \frac{3H\delta}{2\Gamma(\alpha)} + \frac{4H\delta}{\Gamma(\alpha+1)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore by Arzela - Ascoli theorem, $Q : K \to K$ is completely continuous.

Theorem 3.5. Assume that $f(t,x)$ is continuous on $[0,1] \times [0,\infty)$, and satisfies one of the following conditions

$(H1)$ There exist $0 < \eta_1$, $\zeta_1 \leq 1$ such that

$$\lim_{x\to\infty} \frac{f(t,x(t))}{x^{\eta_1}} = 0, \lim_{x\to 0} \frac{f(t,x(t))}{x^{\zeta_1}} = \infty$$

for all t $\in [0,1]$.

$(H2)$ There exist $\eta_2, \zeta_2 \geq 1$ such that

$$\lim_{x\to\infty} \frac{f(t,x(t))}{x^{\eta_2}} = \infty, \lim_{x\to 0} \frac{f(t,x(t))}{x^{\zeta_2}} = 0$$

for all $t \in [0,1]$.

Then the problem (1) has one positive solution.

Proof. It is enough to consider existence of fixed point of operator Q in K. It follows from the above lemma that $Q : K \to K$ is a completely continuous operator. Assume that $(H1)$ holds, then there exist $M_1 > 0, M_2 > 0$, such that for all

$$0 < \epsilon < \left(2\int_0^1 H(s)ds\right)^{-1} \text{ and } \rho > \frac{16}{\int_{1/4}^{3/4} H(s)ds} > 0$$

Then

$$\begin{aligned} &f(t,x(t)) \leq \epsilon x^{\eta_1}, for\ t \in [0,1], x \geq M_1 \\ &f(t,x(t)) > \rho x^{\eta_1}, for\ t \in [0,1], 0 \leq x \leq M_2 \end{aligned}$$

11

So we have

$$f(t,x(t)) \leq \epsilon u^{\eta_1} + c, for t \in [0,1], x \in [0,+\infty)$$

where

$$c = \max_{0 \leq t \leq 1, 0 \leq x \leq M_1} |f(t,x(t)| + 1$$

Let

$$\psi_1 = \{x \in K; \|x\| < R_1\}$$

where $R_1 > \left\{1, 2c \int_0^1 H(s)ds\right\}$. For $x \in \partial\psi_1$, from the Lemma (3.2), we have

$$\begin{aligned} |Qx(t)| &= \int_0^1 G(t,s)f(s,x(s))ds \\ &\leq \int_0^1 H(s)(\epsilon|u|^{\eta_1} + c)ds \\ &\leq \epsilon R_1^{\eta_1} \int_0^1 H(s)ds + c\int_0^1 H(s)ds \\ &\leq \frac{R_1}{2} + \frac{R_1}{2} = R_1 \end{aligned}$$

Hence $\|Qx\| \leq R_1 = \|x\|$
Let $\psi_2 = \{x \in K; \|x\| < R_2\}$
where $0 < R_2 < \{1, M_2\}$, then for $x \in \partial\psi_2$, we obtain

$$\begin{aligned} |Qu(t)| &= \left|\int_0^1 G(t,s)f(s,x(s))\right| \\ &\geq \int_{3/4}^{1/4} G(t,s)f(s,x(s))ds \\ &> \frac{\rho}{4}\int_{1/4}^{3/4} H(s)x(s)^{\zeta_1}ds \\ &\geq \frac{\rho}{16}\int_{1/4}^{3/4} H(s)\left\|u\right\|^{\zeta_1} ds \\ &\geq \frac{\rho}{16}\int_{1/4}^{3/4} H(s)R_2 ds \\ &> R_2 = \|x\| \end{aligned}$$

12

$\|Qx\| \geq R_2 = \|x\|$.
Hence the lemma (2.3) implies that the operator Q has one fixed point $x^*(t) \in \bar{\psi}_1|\psi_2$. Then $x^*(t)$ is one positive solution of the given problem(1).
Similarly assume that $(H2)$ holds, then there exist $S_1 > 0, S_2 > 0$, such that for all

$$0 < \epsilon < \left(\int_0^1 H(s)ds\right) \text{ and } \tau > \left(\frac{\int_{1/4}^{3/4} H(s)ds}{16}\right)^{-1} > 0$$

Then we have

$$f(t,x(t)) > \tau u^{\eta_2}, f\ ort \in [0,1], x \geq S_1$$
$$f(t,x(t)) \leq \epsilon x^{\zeta_2}, for\ t \in [0,1], 0 \leq x \leq S_2$$

Let

$$\psi_1 = \{x \in K; \|x\| < R_1\}\ and$$
$$\psi_2 = \{x \in K; \|x\| < R_2\}$$

where $R_1 > \{1, 4S_1\}$, $0 < R_2 < \{1, S_2\}$. Then we have

$$\begin{aligned}|Qx(t)| &\geq \left|\int_{1/4}^{3/4} G(t,s)f(s,x(s))ds\right| \\ &\geq \frac{\tau}{16}\int_{1/4}^{3/4} H(s)\|x\|^{\eta_2} ds \\ &\geq \frac{\tau}{16}\int_{1/4}^{3/4} H(s)\|x\| ds \\ &> R_1 = \|x\|\end{aligned}$$

for $x \in \partial\psi_2$, we have

$$\begin{aligned}|Qx(t)| &\leq \int_0^1 H(s)\epsilon\|x\|^{\zeta_2} ds \\ &\leq \epsilon R_2 \int_0^1 H(s)ds \\ &\leq R_2\end{aligned}$$

Then the lemma (2.3) implies that the operator Q has one fixed point $x^*(t) \in \bar{\psi}_1|\psi_2$, Hence $x^*(t)$ is a positive solution of the given problem.
Hence the proof.

13

IV. ILLUSTRATIVE EXAMPLE

In this section we provide an example to illustarte the execution of the main result Consider the problem

$$D_{0+}^{\alpha}x(t) = \frac{1}{12^{\alpha}}\left(\frac{1}{1+x^2}\right)cos(2\pi t), t \in [0,1], 1 < \alpha \leq 2$$

Consider

$$f(t,x(t)) = \left(\frac{1}{1+x^2}\right) \tag{13}$$

Now apply (13) in $(H1)$ and $(H2)$ we get

$$\begin{aligned}\lim_{x\to\infty}\frac{f(t,x(t))}{x} &= \lim_{x\to\infty}\frac{1}{(1+x^2)x}\\ &= \lim_{x\to\infty}\frac{1}{x^2(\frac{1}{x^2}+1)x} = 0\end{aligned}$$

similarly

$$\begin{aligned}\lim_{x\to 0}\frac{f(t,x(t))}{x} &= \lim_{x\to 0}\frac{1}{(1+x^2)x}\\ &= \lim_{x\to 0}\frac{1}{x^2(\frac{1}{x^2}+1)x} = \infty\end{aligned}$$

Hence the solution.

ACKNOWLEDGMENT

The first author thanks University Grants Commission (UGC), India for the financial support under UGC Minor Research Project (File No: MRP - 5588/15 (SERO/UGC) dated January 2015.

REFERENCES

[1] O. P. Agrawal, "Formulation of Euler-Larange equations for fractional variational problems", *J. Math. Anal. Appl.* 272, 368-379, 2002.

[2] Bashmir Ahmed - Juan. J. Nieto, "Existence of Solutions for Antiperiodic Boundary Value Problems Involving Fractional Differential Equations Via Leray - Schander Degree Theory", *Mathematics Subject Classification*, 34A12, 34A40, 2010.

[3] D. Delbosco and L.Rodino," Existence and Uniqueness for a Nonlinear Fractional Differential Equation", *J. Math. Appl*, 204, 609-625, 1996.

[4] Krasnosel'skii M. A., " Positive Solutions Of Operator Equations", *Noordhoof Gronigen*, Netherland, 1964.

[5] R. W. Leggett, L. R. Williams," Multiple positive solutions of nonlinear operators on ordered Banach spaces", *Indiana Univ.Math.J.*28(1979) 673-688.

[6] I. Podlubny," Fractional Differential equations", *Mathematics in Science and Engineering*, vol, 198, Academic Press, New Tork/Londin/Toronto, 1999.

[7] S. G. Samko, A. A. Kilbas, O. I. Marichev, "Fractional Integral And Derivatives (Theory and Applications)", *Gordan and Breach.* Switzerland, 1993.

[8] Shuqin Zhang, " Positive Solutions For Boundary Value Problems Of Nonlinear Fractional Differential E quations", *Electronic Journal of Differential Equations*, Vol. 2006(2006), NO. 36, pp. 1 - 12.

[9] Shu-qin Zhang," The Existence of a Positive Solution for a Nonlinear Fractional Differential Equation", *J. Math. Anal. Appl.* 252, 804-812, 2000.

[10] Shu-qin Zhang," Existence of Positive Solution for some class of Nonlinear Fractional Differential Equations", *J. Math. Anal. Appl.* 278, 1, 136-148, 2003.

[11] Zhanbing Bai, Haishen Lu, " Positive solutions For Boundary Value Problem Of Nonlineaf Fractional Differential Equation", *J. Math. Anal. Appl.* 311, 495 - 505, 2005.